The Locus of Curves with D_n-Symmetry inside $\mathfrak{M}_g$

Binru Li, Sascha Weigl

Abstract

The aim of this paper is to determine the irreducible components of $\mathfrak{M}_g(D_n)$, the locus inside $\mathfrak{M}_g$ of the curves admitting an effective action by the dihedral group D_n. This is done by classifying pairs (H, H') of distinct subgroups of the mapping class group Map_g, such that both H and H' are isomorphic to D_n and the fixed point locus of H inside the Teichmüller space $\mathcal{T}_g$ is contained in the fixed point locus of H'.

1 Introduction

Given a finite group H, denote by $\mathfrak{M}_g(H)$ the locus inside $\mathfrak{M}_g$ (the coarse moduli space of curves of genus $g \geq 2$) of the curves admitting an effective action by the group H. A good approach to understanding the irreducible components of $\mathfrak{M}_g(H)$ is to view $\mathfrak{M}_g$ as the quotient of the Teichmüller space $\mathcal{T}_g$ by the natural action of the mapping class group Map_g:

$$\pi : \mathcal{T}_g \to \mathcal{T}_g / Map_g = \mathfrak{M}_g.$$

Observe that

$$\mathfrak{M}_g(H) = \bigcup_{[\rho]} \mathfrak{M}_{g,\,\rho}(H),$$

where $\rho : H \hookrightarrow Map_g$ is an injective homomorphism, $\mathfrak{M}_{g,\,\rho}(H)$ is the image of the fixed locus of $\rho(H)$ under the natural projection π and $\rho \sim \rho'$ iff they are equivalent by the equivalence relation generated by the automorphisms of H and the conjugations by Map_g. We call this equivalence class an *unmarked topological type* (cf. [CLP2], section 2). Since each $\mathfrak{M}_{g,\,\rho}(H)$ is an irreducible (Zariski) closed subset of $\mathfrak{M}_g$ (cf. [CLP2], Theorem 2.3), in order to determine the irreducible components of $\mathfrak{M}_g(H)$, it suffices to determine the maximal loci of the form $\mathfrak{M}_{g,\,\rho}(H)$, i.e. to figure out when one locus contains another.

The case where H is a cyclic group was investigated in [Cor] and [Cat1]. In [CLP2] the authors have defined a new homological invariant which allows them to tell when two homomorphism ρ and ρ' are not equivalent; for the case of $H = D_n$, the dihedral group, they also found one

representative for each unmarked topological type.
In this paper, we focus on the case $H = D_n$, and solve the following problem: for which ρ and ρ', does $\mathfrak{M}_{g,\,\rho'}(D_n)$ contain $\mathfrak{M}_{g,\,\rho}(D_n)$? Hence we determine the loci $\mathfrak{M}_{g,\,\rho}(D_n)$ which are not maximal whence the irreducible decomposition of $\mathfrak{M}_g(D_n)$. The above problem is equivalent to the classification of subgroups H, H' of Map_g ($g \geq 2$), where H and H' satisfy the following condition:

$$(*)\ H, H' \simeq D_n, H \neq H' \text{ and } Fix(H) \subset Fix(H').$$

For any finite subgroup $H \subset Map_g$, set $\delta_H :=$dim$Fix(H)$ and let $G := G(H) := \bigcap_{C \in Fix(H)} Aut(C)$ ($Fix(H)$ corresponds to the complex structures for which the action of H is holomorphic, whereas $G(H)$ is the common automorphism group of all the curves in $Fix(H)$). If $H = G(H)$ we call H *full*.
It is easy to see that condition $(*)$ is equivalent to the condition
$(**)$ H is isomorphic to D_n and not full, $G(H)$ has a subgroup H' which is isomorphic to D_n and different from H.

For any curve $C \in Fix(G)$, we have a Galois cover $p : C \to C/G =: C'$ which is branched in r (r can be zero) points $P_1, ..., P_r$ on C' with branching indices $m_1, ..., m_r$. By Theroem 3.1, in our case C' is always $\mathbb{P}^1$. The cover map p is determined by a surjective homomorphism f from the orbifold fundamental group $T(m_1, ..., m_r) :=< \gamma_1, ..., \gamma_r | \Pi\gamma_i = 1, \gamma_i^{m_i} = 1 >$ to G (cf. [Cat2], section 5). The vector $v := (f(\gamma_1), ..., f(\gamma_r))$ is called the *Hurwitz vector* associated to f (See section 5 for more details). Then two Hurwitz vectors v and v' determine the same topological type if and only if they are equivalent for the equivalence relation generated by the action of $Aut(G)$ and by sequences of braid moves. (See Definition 5.1).

Our main result is the following:

Theorem. *Let H, H' be subgroups of Map_g, satisfying condition $(*)$. Then $G(H) \simeq D_n \times \mathbb{Z}/2$ and H corresponds to $D_n \times \{0\}$. The group H' and the topological action of the group $G(H)$ (i.e. its Hurwitz vector) are as listed in the tables of section 2.*

The structure of this paper is as follows:

In section 2 we present our results through tables.
In section 3 we quote a Theorem from [MSSV] (cf. Theorem 3.1), which contains the possible cases (which we call *cover type*) where $H \subsetneq G \subset Map_g$ and $\delta_G = \delta_H$. From this Theorem, using the Riemann-Hurwitz formula, we obtain pairs of dimensions $(\delta_H, \delta_{H'})$, which can occur under condition (**). We will also see that $C/G \simeq \mathbb{P}^1$ and $[G : H] = 2$ except for one case.

In section 4 we will understand group theoretically which cases of H and G can happen under condition (**). This is done by classifying the index 2 subgroups of G, where G is a finite group containing two distinct index 2 subgroups which are isomorphic to D_n. The cases there are called the *group types*.
In section 5 we classify the equivalence classes of Hurwitz vectors of the map $C \to C/G \simeq \mathbb{P}^1$ for each cover type and group type, by giving one representative vector for each equivalence class.

2 Results

We present our results through tables. There will be one table for each normal form of Hurwitz vectors for the covering $C \to C/G$, obtained in section 5. For the reader's convenience we present a short list of notation:

v	Hurwitz vector for the covering $C \to C/G$
$v_{G/H'}$	Hurwitz vector for the double covering $C/H' \to C/G = \mathbb{P}^1$
$g_{C/H'}$	Genus of C/H'
$\delta_{H'}$	Dimension of $Fix(H')$
$v_{H'}$	Hurwitz vector for the covering $C \to C/H'$

We will use the following subgroups of $D_n \times \mathbb{Z}/2$, where $D_n =< x, y \mid x^n = y^2 = 1, yxy^{-1} = x^{-1} >$ and e denotes the neutral element of D_n.

Subgroup	Generators
K	$(x, 0)$
$H_{1,1}$	$K, (e, 1)$
$H_{1,2}$	$K, (y, 1)$
$H_{1,3}$	$(x^2, 0), (y, 0), (e, 1)$
$H_{1,4}$	$(x^2, 0), (y, 0), (x, 1)$
$H_{1,5}$	$(x^2, 0), (yx, 0), (e, 1)$
$H_{1,6}$	$(x^2, 0), (yx, 0), (x, 1)$

For compactness, we make the following conventions:
Whenever the groups $H_{1,4}, H_{1,6}, H_{1,3}, H_{1,5}$ occur, we assume that $n = 2m$, in the last 2 cases we additionally assume m to be odd. If $H_{1,1}$ appears we are in the case $n = 2$. We identify the groups $H_{1,3}$ and $H_{1,5}$ with D_n by sending their respective generators in the given order to x^{m+1}, y, x^m.
The cover types are those which appear in Theorem 3.1.

Theorem 2.1. *Let H, H' be subgroups of $Mapg_g$, satisfying condition $(*)$. Then $G(H) \simeq D_n \times \mathbb{Z}/2$, H corresponds to $D_n \times \{0\}$. The group H' and the topological action of the group $G(H)$ (i.e. its Hurwitz vector) are as listed in the following tables.*

We obtain immediately the following corollary:

Corollary 2.2. *The locus $\mathfrak{M}_{g,\,\rho}(D_n)$ is maximal iff its topological type $[\rho]$ is different from those which are determined by $C \to C/H$ in the following tables.*

Remark 2.3. *Given a cover $C \to C/H$, the data consisting of $g_{C/H}$ and the branching indices are called the signature of the cover. In* [BCGG]*, section 3 the authors computed the signatures for the possible non-maximal loci of the form $\mathfrak{M}_{g,\,\rho}(D_n)$, which is a corollary of our result.*

Cover type I)

($\delta_H = 3$, $g_{C/H} = 2$, $C \to C/H$ is unramified)

$$v = (((y,1),(y,1),(yx,1),(yx,1),(e,1),(e,1))$$

H'	$v_{G/H'}$	$g_{C/H'}$	$\delta_{H'}$	$v_{H'}$
$H_{1,2}$	(0,0,0,0,1,1)	0	5	(y,y,y,y,yx,yx,yx,yx)
$H_{1,3}$	(0,0,1,1,0,0)	0	5	$(yx^m, yx^{m-2}, yx^m, yx^{m+2}, x^m, x^m, x^m, x^m)$
$H_{1,4}$	(1,1,0,0,1,1)	0	5	(yx,yx,yx,yx,y,y,y,y)
$H_{1,5}$	(1,1,0,0,0,0)	0	5	$(yx^m, yx^m, yx^m, yx^m, x^m, x^m, x^m, x^m)$
$H_{1,6}$	(0,0,1,1,1,1)	1	4	(e,yx;y,y,y,y)

$$v = ((y,1),(yx^m,1),(yx,1),(yx,1),(x^m,1),(e,1)),\ \ n = 2m$$

H'	$v_{G/H'}$	$g_{C/H'}$	$\delta_{H'}$	$v_{H'}$
$H_{1,2}$	(0, 0, 0, 0, 1, 1)	0	5	$(y, y, yx^m, x^m y, yx, yx, yx, yx)$
$H_{1,3}$	(0, 1, 1, 1, 1, 0)	1	4	$(x^{m+1}, x^{m-1}; x^m, x^m, yx^m, yx^m)$
$H_{1,4}$(m odd)	(1, 0, 0, 0, 0, 1)	0	5	$(yx^m, x^m y, yx, yx^3, yx, xy, x^m, x^m)$
$H_{1,4}$(m even)	(1, 1, 0, 0, 1, 1)	1	4	$(x^m, x^m y; yx, yx, yx, yx)$
$H_{1,5}$	(1, 0, 0, 0, 1, 0)	0	5	$(yx^{\frac{m^2-1}{2}}, yx^{\frac{m^2-1}{2}}, yx^m, yx^m, yx^m, yx^m, x^m, x^m)$
$H_{1,6}$(m odd)	(0,1,1,1,0,1)	1	4	$(x^{m+1}, x^{m-1}; x^m, x^m, y, y)$
$H_{1,6}$(m even)	(0,0,1,1,1,1)	1	4	$(e, x^{m-1}y; y, y, x^m y, yx^m)$

$$v = ((y, 1), (yx^m, 1), (yx^2, 1), (yx^2, 1), (x^m, 1), (e, 1)),\ \ n = 2m,\ \ m \text{ odd}.$$

H'	$v_{G/H'}$	$g_{C/H'}$	$\delta_{H'}$	$v_{H'}$
$H_{1,2}$	$(0, 0, 0, 0, 1, 1)$	0	5	$(y, y, yx^m, yx^m, yx^2, yx^2, yx^2, yx^2)$
$H_{1,3}$	$(0, 1, 0, 0, 1, 0)$	0	5	$(x^m, x^m, yx^m, yx^{-1}, yx^{-3}, yx^{-1}, yx)$
$H_{1,4}$	$(0, 0, 0, 0, 1, 1)$	0	5	$(y, y, yx^m, yx^m, yx^2, yx^2, yx^2, yx^2)$
$H_{1,5}$	$(1, 0, 1, 1, 1, 0)$	1	4	$(x^2, x^{-2}; x^m, x^m, x^m y, x^m y)$
$H_{1,6}$	(0,1,0,0,0,1)	0	5	$(y, y, x^2 y, x^6 y, x^2 y, yx^2, x^m, x^m)$

For $n = 2$ we have two extra cases:

$$v = ((y, 1), (y, 1), (x, 1), (x, 1), (e, 1), (e, 1))$$

H'	$v_{G/H'}$	$g_{C/H'}$	$\delta_{H'}$	$v_{H'}$
$H_{1,1}$	$(1, 1, 0, 0, 0, 0)$	0	5	(x, x, x, x, y, y, y, y)
$H_{1,2}$	$(0, 0, 1, 1, 1, 1)$	1	4	$(e, x; y, y, y, y)$
$H_{1,3}$	$(0, 0, 1, 1, 0, 0)$	0	5	$(yx, yx, yx, yx, x, x, x, x)$
$H_{1,4}$	$(1, 1, 0, 0, 1, 1)$	1	4	$(e, y; x, x, x, x)$
$H_{1,5}$	$(1, 1, 1, 1, 0, 0)$	1	4	$(e, yx; x, x, x, x)$
$H_{1,6}$	(0,0,0,0,1,1)	0	5	(y, y, y, y, x, x, x, x)

$$v = ((y, 1), (yx, 1), (x, 1), (x, 1), (x, 1), (e, 1))$$

H'	$v_{G/H'}$	$g_{C/H'}$	$\delta_{H'}$	$v_{H'}$
$H_{1,1}$	$(1, 1, 0, 0, 0, 0)$	0	5	$(yx, yx, yx, yx, yx, yx, y, y)$
$H_{1,2}$	$(0, 0, 1, 1, 1, 1)$	1	4	$(e, e; y, y, yx, yx)$
$H_{1,3}$	$(0, 1, 1, 1, 1, 0)$	1	4	$(y, y; x, x, yx, yx)$
$H_{1,4}$	$(1, 0, 0, 0, 0, 1)$	0	5	$(yx, yx, x, x, x, x, x, x, x, x)$
$H_{1,5}$	$(1, 0, 1, 1, 1, 0)$	1	4	$(e, y; x, x, x, x)$
$H_{1,6}$	(0,1,0,0,0,1)	0	5	$(y, y, x, x, x, x, x, x, x, x)$

Cover type II)

$(\delta_H = 2, g_{C/H} = 1)$

(1) $c_5 = 2$.

$$v = ((y,1),(yx,1),(yx,1),(e,1),(y,0)),\ \ v_H = (x,x^{-1};y,y).$$

H'	$v_{G/H'}$	$g_{C/H'}$	$\delta_{H'}$	$v_{H'}$
$H_{1,2}$	$(0,0,0,1,1)$	0	3	(y,y,yx,yx,yx,yx)
$H_{1,3}$	$(0,1,1,0,0)$	0	3	$(yx^m,yx^{-1},x^m,x^m,y,yx^{m+1})$
$H_{1,4}$	$(1,0,0,0,1)$	0	3	(yx,yx,yx,yx,y,y)
$H_{1,5}$	$(1,0,0,0,1)$	0	3	$(yx^m,yx,yx^m,yx^{-1},x^m,x^m)$
$H_{1,6}$	$(0,1,1,1,1)$	1	2	$(e,yx;y,y)$

(2) $c_5 > 2$.

$$v = ((y,1),(yx^{-1},1),(e,1),(e,1),(x,0)),\ \ c_5 = n,\ \ v_H = (x^{-1},y;x,x).$$

H'	$v_{G/H'}$	$g_{C/H'}$	$\delta_{H'}$	$v_{H'}$
$H_{1,2}$	$(0,0,1,1,0)$	0	3	$(y,y,yx^{-1},yx^{-3},x,x)$
$H_{1,3}$	$(0,1,0,0,1)$	0	4	$(yx^m,yx^{-1},x^m,x^m,x^m,x^m,x^{m+1})$
$H_{1,4}$	$(1,0,1,1,1)$	1	3	$(y,y;x^2,yx^3,yx)$
$H_{1,5}$	$(1,0,0,0,1)$	0	4	$(yx^{-1},yx^{m-2},x^m,x^m,x^m,x^m,x^{m+1})$
$H_{1,6}$	$(0,1,1,1,1)$	1	3	$(yx^{-1},yx^{-1};x^2,yx^2,y)$

$$v = ((y,1),(yx^{m-1},1),(x^m,1),(e,1),(x,0)),\ \ n = 2m,\ \ v_H = (x^{m-1},yx^m;x,x).$$

H'	$v_{G/H'}$	$g_{C/H'}$	$\delta_{H'}$	$v_{H'}$
$H_{1,2}$	$(0,0,1,1,0)$	0	3	$(y,y,yx^{m-1},yx^{m-3},x,x)$
$H_{1,3}$	$(0,0,1,0,1)$	0	4	$(yx^m,yx^{m+2},yx,yx,x^m,x^m,x^{-2})$
$H_{1,4}$ (m odd)	$(1,1,0,1,1)$	1	3	$(x^{m-1},y;x^2,x^m,x^m)$
$H_{1,4}$ (m even)	$(1,0,1,1,1)$	1	3	$(yx^m,y;x^2,yx^{m+3},yx^{m+1})$
$H_{1,5}$	$(1,1,1,0,1)$	1	3	$(x^{m+1},y;x^{-2},x^m,x^m)$
$H_{1,6}$ (m odd)	$(0,0,0,1,1)$	0	4	$(y,yx^{-2},yx^{m-1},yx^{m-1},x^m,x^m,x^2)$
$H_{1,6}$ (m even)	$(0,1,1,1,1)$	1	3	$(yx^{-1},yx^{m-1};x^2,yx^2,y)$

$$v = ((y,1),(yx^{m-2},1),(x^m,1),(e,1),(x^2,0)),\ \ n = 2m,\ \ m \text{ odd},\ \ c_5 = m,$$

$$v_H = (x^{m-2}, yx^m; x^2, x^2),$$

H'	$v_{G/H'}$	$g_{C/H'}$	$\delta_{H'}$	$v_{H'}$
$H_{1,2}$	$(0,0,1,1,0)$	0	3	$(y,y,yx^{m-2},yx^{m-6},x^2,x^2)$
$H_{1,3}$	$(0,1,1,0,0)$	0	3	$(yx^m,yx^{m-4},x^m,x^m,x^2,x^2)$
$H_{1,4}$	$(1,0,0,1,0)$	0	3	$(yx^{m-2},yx^{m-6},x^m,x^m,x^2,x^2)$
$H_{1,5}$	$(1,0,1,0,0)$	0	3	$(y,yx^{-4},x^m,x^m,x^2,x^2)$
$H_{1,6}$	$(0,1,0,1,0)$	0	3	$(y,yx^{-4},x^m,x^m,x^2,x^2)$

For $n = 2$ we have one extra case.

$$v = ((yx,1),(x,1),(e,1),(e,1),(y,0)),\ \ v_H = (y, yx; y, y).$$

H'	$v_{G/H'}$	$g_{C/H'}$	$\delta_{H'}$	$v_{H'}$
$H_{1,1}$	$(1,0,0,0,1)$	0	3	(x,x,y,y,y,y)
$H_{1,2}$	$(0,1,1,1,1)$	1	2	$(x,x;yx,yx)$
$H_{1,3}$	$(1,1,0,0,0)$	0	3	(x,x,x,x,y,y)
$H_{1,4}$	$(0,0,1,1,0)$	0	3	(yx,yx,x,x,y,y)
$H_{1,5}$	$(0,1,0,0,1)$	0	3	(yx,yx,x,x,x,x)
$H_{1,6}$	$(1,0,1,1,1)$	1	2	$(yx,yx;x,x)$

Cover type III-a)

$(\delta_H = 1,\ g_{C/H} = 1)$

$$v = ((y,1),(yx^{-1},1),(e,1),(x,1)),\ \ 2d_4 = n = 2m,\ \ v_H = (x^{-1}, y; x^2).$$

H'	$v_{G/H'}$	$g_{C/H'}$	$\delta_{H'}$	$v_{H'}$
$H_{1,2}$	$(0,0,1,1)$	0	2	$(y,yx^2,yx^{-1},yx^{-1},x^2)$
$H_{1,3}$	$(0,1,0,1)$	0	2	$(yx^m,yx^{-1},x^m,x^m,x^{m+1})$
$H_{1,4}$	$(1,0,1,0)$	0	1	(yx^{-1},yx^{-3},x,x)
$H_{1,5}$	$(1,0,0,1)$	0	2	$(yx^{-1},yx^{m-2},x^m,x^m,x^{m+1})$
$H_{1,6}$	$(0,1,1,0)$	0	1	(x,x,y,yx^{-2})

Cover type III-b)
($\delta_H = 1$, $g_{C/H} = 0$)

$$v = ((yx,1),(e,1),(y,0),(x,0)),\ \ c_4 = n = 2m,\ \ v_H = (y, yx^{-2}, x, x).$$

H'	$v_{G/H'}$	$g_{C/H'}$	$\delta_{H'}$	$v_{H'}$
$H_{1,2}$	$(0,1,1,0)$	0	1	(yx, yx^{-1}, x, x)
$H_{1,3}$	$(1,0,0,1)$	0	2	$(x^m, x^m, y, yx^{m-1}, x^{m+1})$
$H_{1,4}$	$(0,1,0,1)$	0	2	(yx, yx^{-1}, y, y, x^2)
$H_{1,5}$	$(0,0,1,1)$	0	2	$(yx^m, yx^{-1}, x^m, x^m, x^{m+1})$
$H_{1,6}$	$(1,1,1,1)$	1	1	$(yx, x; x^2)$

3 A rough classification

In this section we determine the possible pairs of dimensions $(\delta_H, \delta_{H'})$, for distinct subgroups H and H' of Map_g which satisfy condition (**).
Given $C \in Fix(H)$, assume that $C \to C/H$ is a cover branched on r points. We have that $\delta_H = 3g_{G/H} - 3 + r$ (cf. [CLP2], Theorem 2.3).
The case $\delta_H = \delta_{H'}$ was done in Corollary 7.2 of [CLP2]. We only consider the case $\delta_H < \delta_{H'}$.
We recall Lemma 4.1 of [MSSV]:

Theorem 3.1. *(MSSV)*
Let $H \subsetneq G$ be two (finite) subgroups of Map_g, $\delta_H = \delta_G =: \delta$. Then one of the following holds:
$I)$ $\delta_H = 3$, $[G{:}H] = 2$, $C \to C/G$ is a covering of $\mathbb{P}^1$ branched on 6 points $P_1, \ldots, P_6$, and with branching indices all equal to 2. Moreover the subgroup H corresponds to the unique genus two double cover of $\mathbb{P}^1$ branched on the 6 points.
$II)$ $\delta_H = 2$, $[G{:}H] = 2$, and $C \to C/G$ is a covering of $\mathbb{P}^1$ branched on five points, $P_1, \ldots, P_5$, with branching indices $2,2,2,2,c_5$. Moreover the subgroup H corresponds to a double cover of $\mathbb{P}^1$ branched on the 4 points $P_1, \ldots, P_4$ with branching index 2.
$III)$ $\delta_H = 1$, there are 3 possibilities:
$III-a)$ H has index 2 in G, and $C \to C/G$ is a covering of $\mathbb{P}^1$ branched on 4 points, $P_1, \ldots, P_4$, with branching indices $2,2,2,2d_4$, where $d_4 > 1$. Moreover the subgroup H corresponds to the unique genus one double cover of $\mathbb{P}^1$ branched on the 4 points $P_1, \ldots, P_4$.
$III-b)$ H has index 2 in G, and $C \to C/G$ is a covering of $\mathbb{P}^1$ branched on 4 points, $P_1, \ldots, P_4$, with branching indices $2,2,c_3,c_4$, where $c_3 \le c_4$ and $c_4 > 2$. Moreover the subgroup H corresponds to a genus zero double cover of $\mathbb{P}^1$ branched on two points with branching index 2.
$III-c)$ H is normal in G, $G/H \cong (\mathbb{Z}/2)^2$, moreover $C \to C/G$ is a covering of $\mathbb{P}^1$ branched on

4 points $P_1, \dots, P_4$, with branching indices $2, 2, 2, c_4$, where $c_4 > 2$. Moreover the subgroup H corresponds to the unique genus zero cover of $\mathbb{P}^1$ with group $(\mathbb{Z}/2)^2$ branched on the 3 points P_1, P_2, P_3 with branching index 2.

We call the cases in Theorem 3.1 the *cover type* (of H and G).
Since we have condition $(**)$, which implies $\delta_G = \delta_H$, we can apply Theorem 3.1. Moreover we apply the Riemann-Hurwitz formula to each cover type to find the possible pairs $(\delta_H, \delta_{H'})$.

Corollary 3.2. *Assume $(**)$ and moreover $\delta_H < \delta_{H'}$. Then the following pairs of dimensions $(\delta_H, \delta_{H'})$ can occur:*

I) $(3, 4)$, $(3, 5)$.
II) $(2, 3)$, $(2, 4)$.
$III - a$) $(1, 2)$.
$III - b$) $(1, 2), (1, 3)$.
$III - c$) *None.*

Proof. I) $\delta_H = 3$.
By the Riemann-Hurwitz formula,

$$2g(C) - 2 = |G|(-2 + 6 \cdot \frac{1}{2}) = |H'|(2(g_{C/H'} - 1) + k/2)$$

where k is the number of branching points of $C \to C/H'$.
It is easy to see that $(g_{C/H'}, k) = (2, 0), (1, 4)$ or $(0, 8)$, corresponding to $\delta_{H'} = 3, 4, 5$. Since we require $\delta_H < \delta_{H'}$, the possible pairs are (3,4) and (3,5).
II) $\delta_H = 2$.
In this case $C/H' \to \mathbb{P}^1$ is a double covering branched on at most 5 points. Using Riemann-Hurwitz, there are two cases:
(i) $g_{C/H'} = 0$ and $C/H' \to \mathbb{P}^1$ is branched on 2 of the 5 points with branching indices 2,2.
If $c_5 = 2$ or P_5 is not a branching point, we have $\delta_{H'} = 3$;
Otherwise c_5 is even and bigger than 2 and P_5 is a branching point, we get $\delta_{H'} = 4$.
(ii) $g_{C/H'} = 1$ and $C/H' \to \mathbb{P}^1$ is branched on 4 of the 5 points with branching indices 2,2,2,2.
The only possible case in which $\delta_{H'} > 2$ is that c_5 is even and bigger than 2 and P_5 is one of the branching points. In this case $\delta_{H'} = 3$.
III) $\delta_H = 1$.
$III - a$) Similar to case II), one gets $g_{C/H'} = 0$, and $C/H' \to \mathbb{P}^1$ is a double cover with one of the branching points P_4 and $\delta_{H'} = 2$.
$III - b$) i) If $c_3 = 2$, the only possibility is c_4 even, $g_{C/H'} = 0$ and $C/H' \to \mathbb{P}^1$ is a double cover

with one of the branching points P_4, here $\delta_{H'} = 2$.
ii) $c_3 > 2$, there are three possibilities:
α) c_3 or c_4 is even, one and only one point of P_3,P_4 is a branching point. This case is similar to $III - b) - i)$, $\delta_{H'} = 2$.
β) Both c_3 and c_4 are even, $g_{C/H'} = 0$, and $C/H' \to \mathbb{P}^1$ is a double cover branching on P_3,P_4. We have $\delta_{H'} = 3$.
γ) Both c_3 and c_4 are even, $g_{C/H'} = 1$, and $C/H' \to \mathbb{P}^1$ is a double cover branching on 4 points $P_1, \ldots, P_4$. We have $\delta_{H'} = 2$.
$III - c$) We will give the proof in section 5, Lemma 5.8. □

Remark: Cor. 3.2 is valid for any H, H' with the same index in G except for the case $III - c$).

4 Index 2 subgroups of G

From Theorem 3.1 we know that $[G{:}H] = 2$ except for $III - c$). Such a pair is given by an exact sequence

$$1 \to H \to G \to \mathbb{Z}/2 \to 1.$$

This type of extensions, where $H = D_n$ and G has another subgroup H' isomorphic to D_n, has been classified in [CLP2], Proposition 7.4. There are 3 cases, which we call *group types*:

Group type 1) $G \cong D_n \times \mathbb{Z}/2$, H corresponds to the subgroup $D_n \times \{0\}$.
Group type 2) $n = 2d$, $G \cong D_{2n} =< z, y | z^{2n} = y^2 = 1, yzy = z^{-1} >$, $H =< x := z^2, y >$.
Group type 3) $n = 4h$, where h is odd, and G is the semidirect product of $H \cong D_n$ with $< \beta_2 > \cong \mathbb{Z}/2$, such that conjugation by β_2 acts as follows:

$$y \mapsto yx^2, x \mapsto x^{2h-1}.$$

For each group type, we will determine the index 2 subgroups of G and find out which of them are isomorphic to D_n.

Group type 1) Recall the standard presentation $D_n =< x, y | x^n = y^2 = 1, yxy^{-1} = x^{-1} >$ and let $C_n := \mathbb{Z}/n$.
We have to understand the index 2 subgroups K of D_n, such that $K \triangleleft G$, where K corresponds to $H \cap H'$.
a) $K = C_n \times 0$ (This is the only case when n is odd).
Since $G/K \cong (\mathbb{Z}/2)^2$, there are two more index 2 subgroups $H_{1,1} :=< K, (e, 1) >$,

$H_{1,2} :=< K, (y, 1) >\cong D_n$.
$b)$ If $n = 2d$, there are two more cases, $K =< (x^2, 0), (y, 0) >$ or $K =< (x^2, 0), (yx, 0) >$ (both isomorphic to D_d).
Here we have 4 more index 2 subgroups, $H_{1,3} :=< (x^2, 0), (y, 0), (e, 1) >$, $H_{1,4} :=< (x^2, 0), (y, 0), (x, 1) >$, $H_{1,5} :=< (x^2, 0), (yx, 0), (e, 1) >$, $H_{1,6} :=< (x^2, 0), (yx, 0), (x, 1) >$. On checks easily that $H_{1,4}$ and $H_{1,6}$ are isomorphic to D_n and that $H_{1,3}$ and $H_{1,5}$ are isomorphic to D_n if and only if d is odd.
Group type 2) Using similar arguments as for group type 1), we obtain 2 more index 2 subgroups: $H_{2,1} = C_{2n}$, $H_{2,2} =< z^2, yz >\cong D_n$.
Group type 3) There are 6 more index 2 subgroups: $H_{3,1} =< C_n, (e, \beta_2) >$, $H_{3,2} =< C_n, (y, \beta_2) >$, $H_{3,3} =< (x^2, 0), (y, 0), (e, \beta_2) >$, $H_{3,4} =< (x^2, 0), (y, 0), (x, \beta_2) >$, $H_{3,5} =< (x^2, 0), (yx, 0), (e, \beta_2) >$, $H_{3,6} =< (x^2, 0), (yx, 0), (x, \beta_2) >$, and only $H_{3,3}$ is isomorphic to D_n (since $H_{3,3} =< (y, \beta_2), (e, \beta_2) >$).

5 Hurwitz vectors for $C \to C/G$

We start by recalling some general theory of Galois covers of Riemann surfaces (cf. [Cat2], section 5).
Let H be a finite group (not necessarily isomorphic to D_n) which acts effectively on a curve C of genus $g \geq 2$, we obtain a Galois cover $p : C \to C/H := C'$ branched on r points with branching indices $m_1, ..., m_r$. Denote by g' the genus of C', the *orbifold fundamental group* of the cover is a group with the following presentation:

$$T(g'; m_1, ..., m_r) :=< \alpha_1, \beta_1, ..., \alpha_{g'}, \beta_{g'}; \gamma_1, ..., \gamma_r \mid \Pi[\alpha_j, \beta_j] \cdot \Pi\gamma_i = 1, \gamma_i^{m_i} = 1 >$$

The cover $C \to C/H$ is (topologically) determined by a surjective morphsim

$$f : T(g'; m_1, ..., m_2) \to G,$$

such that $f(\gamma_j)$ has order m_j inside G. We call $v := [f(\alpha_1), f(\beta_1), ..., f(\alpha_{g'}), f(\beta_{g'}); f(\gamma_1), ..., f(\gamma_r)]$ the *Hurwitz vector* associated to f.
In this section we study the Hurwitz vectors of each cover type $C \to C/G$ in Theorem 3.1. Hence we have that $C/G \simeq \mathbb{P}^1$, and we set $T(m_1, ..., m_r) := T(0; m_1, ..., m_r)$.
Given a morphism $f : T(m_1, ..., m_r) \to G$, the Hurwitz vector associated to f is not uniquely determined, since we can choose different presentations for $T(m_1, ..., m_r)$. For instance consider $T(m_1, ..., m_r)$ with the presentation $< \gamma_1, ...\gamma_r | \Pi\gamma_i = 1, \gamma_i^{m_i} = 1 >$, for any $1 \leq k < r$, we have a set of generators $\{\delta_i\}$, where $\delta_i := \alpha_i$ if $i \neq k, k+1$; $\delta_k := \alpha_k \alpha_{k+1} \alpha_k^{-1}$ and $\delta_{k+1} := \alpha_k$, this induces an isomorphism between $T(m_1, ..., m_r)$ and $T(l_1, ..., l_r)$, where $l_i = m_i$ if $i \neq k, k+1$;

$l_k = m_{k+1}$ and $l_{k+1} = m_k$. Different choices of the generators correspond to the following braid group action on the set of Hurwitz vectors.
Recall that Artin's *braid group on r strands* has the presentation

$$\mathcal{B}_r := < \sigma_1, ..., \sigma_{r-1} | \forall 1 \le i \le r-2,\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}; \forall |j-i| \ge 2, \sigma_i\sigma_j = \sigma_j\sigma_i > .$$

The group $\mathfrak{B}_r$ acts on the set of Hurwitz vectors of length r as follows:

$$(v_1, ..., v_i, v_{i+1}, ..., v_r) \overset{\sigma_i}{\mapsto} (v_1, ..., v_i v_{i+1} v_i^{-1}, v_i, ..., v_r).$$

On the other hand, for any $h \in Aut(G)$, we can compose f with h, this induces a $Aut(G)$-action on the set of Hurwitz vectors: given $v = (v_1, ..., v_r)$ a Hurwitz vector, define $h(v) := (h(v_1), ..., h(v_r))$. Since these actions (by $\mathfrak{B}_r$ and by $Aut(G)$) commute, they induce an action of the group $\mathcal{B}_r \times Aut(G)$ on the set of Hurwitz vectors of length r.

Definition 5.1. *Given two G-Hurwitz vectors v, v' of length r, we say that v and v' are equivalent if they are in the same $\mathcal{B}_r \times Aut(G)$-orbit.*

Remark 5.2. *Two Hurwitz vectors v and v' determine the same unmarked topological type iff they are equivalent (cf. [CLP2], section 2).*

Definition 5.3. *Let $C \to C/G \cong \mathbb{P}^1$ be a Galois cover of a given group type and cover type. We call a homomorphism $f : T(m_1, \ldots, m_r) \to G$ <u>admissible</u> if it satisfies the following two conditions:*
(1) f is surjective, $T(m_1, ..., m_r)$ is isomorphic to the orbifold fundamental group of $C \to C/G$ and $f(\gamma_i)$ has order m_i in G.
(2) $f_H := \pi_H \circ f : T(m_1, \ldots, m_r) \to G/H$ corresponds to the cover $C/H \to \mathbb{P}^1$, where $\pi_H : G \to G/H$ is the quotient homomorphism.

Definition 5.4. *Let $f : T(m_1, ...m_r) \to G$ and $f' : T(l_1, ...l_r) \to G$ be admissible for a given cover type and group type. We say f is equivalent to f' if their corresponding Hurwitz vectors are in the same $\mathfrak{B}_r \times Aut(G)_H$-orbit, where $Aut(G)_H$ denotes the subgroup of $Aut(G)$ which leaves H invariant.*

Remark 5.5. *An admissible f determines both the covers $C \to C/G$ and $C \to C/H$, hence we require the equivalence relation to be generated by $\mathfrak{B}_r$ and $Aut(G)_H$. It can happen that two admissible homomorphisms have equivalent Hurwitz vectors, but are not equivalent (cf. Remark 5.15).*

Example 5.6. *Cover type $III - b$) and group type 1) (cf. Corollary 3.2)*

$i)$ $c_3 = 2$, assume n even and $c_4 = n$.
Consider $f : T(2,2,2,c_4) \to D_n \times \mathbb{Z}/2$: $\gamma_1 \mapsto (yx, 1)$, $\gamma_2 \mapsto (e, 1)$, $\gamma_3 \mapsto (y, 0)$, $\gamma_4 \mapsto (x, 0)$.
$\delta_{H_{1,2}} = \delta_{H_{1,6}} = 1$, $\delta_{H_{1,4}} = 2$.
$ii)$ $c_3 > 2$, assume we have an admissible f, it is easy to see that $f(\gamma_3) = (x^{i_3}, 0)$, $f(\gamma_4) = (x^{i_4}, 0)$. $f(\gamma_1), f(\gamma_2) \in \{(yx^k, 1), k \in \mathbb{Z}; (x^{n/2}, 1)(if\ n\ is\ even)\}$. Since $\Pi f(\gamma_i) = 1$, there are only two possibilities:
(a) $f(\gamma_1), f(\gamma_2) = (x^{n/2}, 1)$, which implies $\mathrm{Im}(f) \subset< (x,0),(0,1) >$, a contradiction.
(b) $f(\gamma_1) = (yx^{i_1}, 1)$, $f(\gamma_2) = (yx^{i_2}, 1)$, which implies $\mathrm{Im}(f) \subset< (x,0),(y,1) >$, again a contradiction.

Now we classify all admissible f's for the covering $C \to C/G$, in the following way: For each cover type and group type, we construct all possible Hurwitz vectors according to their branching behavior, as given in Theorem 3.1.

Lemma 5.7. *Group type* 2) *has no admissible f for any cover type.*

Proof. Cover type I)
Assume we have an admissible $f : T(2,2,2,2,2,2) \to D_{2n}$, then $f_H(\gamma_i) = 1$, $i = 1,\dots,6$, which implies that $f(\gamma_i) \in \{yz^{2k+1}, z^{2l+1}, k, l \in \mathbb{Z}\}$. Moreover $f(\gamma_i)$ has order two, thus $f(\gamma_i) \in \{yz^{2k+1}, k \in \mathbb{Z}\}$. We find that $\mathrm{Im}(f) \subset H_{2,2}$, a contradiction.
Cover type II)
If there exists an admissible $f : T(2,2,2,2,c_5) \to D_{2n}$, we get $f(\gamma_i) \in \{yz^{2k+1}, k \in \mathbb{Z}\}, i = 1,2,3,4$ and $f(\gamma_5) \in \{z^{2l}, l \in \mathbb{Z}\}$(since $\Pi f(\gamma_i) = 1$), which implies that $\mathrm{Im}(f) \subset H_{2,2}$, a contradiction.
Cover type III-a)
Given an admissible $f : T(2,2,2,2d_4) \to D_{2n}$, we get $f(\gamma_i) \in \{yz^{2k+1}, k \in \mathbb{Z}\}, i = 1,2,3$, and $f(\gamma_4) \in \{z^{2l+1}, l \in \mathbb{Z}\}$. However, $\Pi f(\gamma_i) \neq 1$, a contradiction.
Cover type III-b)
$i)$ $c_3 = 2$. We have $f(\gamma_i) = yz^{2k_i+1}, i = 1,2$, $f(\gamma_3) = yz^{2k_3}$ or z^n, $f(\gamma_4) = z^{2k_4}$. If $f(\gamma_3) = yz^{2k_3}$ we find $\Pi f(\gamma_i) \neq 1$; otherwise $f(\gamma_3) = z^n$, which implies $\mathrm{Im}(f) \subset< yz, z^2 >$. In both cases we have no admissible f.
$ii)$ $c_3 > 2$. We have $(f(\gamma_1), f(\gamma_2), f(\gamma_3), f(\gamma_4)) = (yz^{2k_1+1}, yz^{2k_2+1}, z^{2k_3}, z^{2k_4})$. We see $\mathrm{Im}(f) \subset< yz, z^2 >$, a contradiction. □

Lemma 5.8. *Group type* 3) *has no admissible f for any cover type.*

Proof. First we determine the order 2 elements of type (a, β_2) in G. One computes easily that

$(x^j, \beta_2)^2 = (x^{2jh}, 0)$ and $(yx^k, \beta_2)^2 = (x^{2kh-2k+2}, 0) \neq (e, 0)$. Therefore we conclude that (a, β_2) is of order two $\Leftrightarrow a = x^j$ and j is even.

Cover type I)

Now assume we have an admissible f, which implies that $f(\gamma_i) = (x^{2j_i}, \beta_2)$. However these elements are contained in the proper subgroup $< (x^2, 0), (e, \beta_2) >$, we see f can not be surjective, a contradiction.

Cover type II)

If there exists an admissible f, we must have $f(\gamma_i) = (x^{2j_i}, \beta_2), i = 1, 2, 3, 4$, and since $\Pi f(\gamma_i) = 1$ it follows that $\mathrm{Im}(f) \subset< (x^2, 0), (e, \beta_2) >$, a contradiction.

Cover type III-a)

Assume we have an admissible f, we see that $f(\gamma_i) = (x^{2j_i}, \beta_2), i = 1, 2, 3$. Since $\Pi f(\gamma_i) = 1$ it follows that $\mathrm{Im}(f) \subset< (x^2, 0), (e, \beta_2) >$, again a contradiction.

Cover type III-b)

i) $c_3 = 2$. We must have $f(\gamma_1) = (x^{2j_1}, \beta_2)$, $f(\gamma_2) = (x^{2j_2}, \beta_2)$, $f(\gamma_3) = (x^{2h}, 0)$ or $(yx^k, 0)$, $f(\gamma_4) = (x^l, 0), l \neq 2h$. If $f(\gamma_3) = (x^{2h}, 0)$, then $\mathrm{Im}(f) \subset< (x, 0), (0, \beta_2) >$; if $f(\gamma_3) = (yx^k, 0)$ we see $\Pi f(\gamma_i) \neq 1$. In both cases we can not get an admissible f.

ii) $c_3 > 2$. Given an admissible f, we have $f(\gamma_1) = (x^{2j_1}, \beta_2)$, $f(\gamma_2) = (x^{2j_2}, \beta_2)$, $f(\gamma_3) = (x^{k_3}, 0)$ and $f(\gamma_4) = (x^{k_4}, 0)$ $(k_3, k_4 \neq 2h)$. One sees immediately that $\mathrm{Im}(f) \subset< (x, 0), (0, \beta_2) >$, a contradiction. ◻

Lemma 5.9. *Cover type* $III - c$) *has no admissible f.*

Proof. Assume that we have an admissible $f : T(2, 2, 2, c_4) \to G$.

Let $(b_1, b_2, b_3, b_4) := (f(\gamma_1), f(\gamma_2), f(\gamma_3), f(\gamma_4))$. We have

(1) $b_1^2 = b_2^2 = b_3^2 = 1$. Since $b_4 \in H$ and $order(b_4) = c_4 > 2$, we see that b_4 must lie in the cyclic group, say $b_4 = x^k$, we also find $n > 2$.

(2) The fact that H is normal in G implies that $b_i x b_i = x^{k_i}, i = 1, 2, 3$, therefore $x^k b_i = b_i x^{kk_i}, i = 1, 2, 3$.

(3) $b_1 b_2 b_3 b_4 = 1 \Rightarrow b_1 b_2 = x^{-k} b_3$, moreover $(b_1 b_2)^2 = x^{-k} b_3 x^{-k} b_3 = x^{-k-kk_3}$.

Any element in $\mathrm{Im}(f)$ has the form $\Pi_{i=1}^{l} \beta_i$, where $\beta_i \in \{x^k, x^{-k}, b_1, b_2, b_3\}$. Since $b_1 b_2 b_3 b_4 = 1$, without loss of generality we can assume $\beta_i \in \{x^k, x^{-k}, b_1, b_2\}$, which means that every element in $\mathrm{Im}(f)$ is a word in these four elements.

Using (2), we can "move" the $x^{\pm k}$ terms to the end. Taking (1) into account, we see that the elements are of the forms $(b_1 b_2)^s x^t$, $b_2 (b_1 b_2)^s x^t$ or $(b_1 b_2)^s b_1 x^t$, now use (3), one sees immediately that elements in $\mathrm{Im}(f)$ have the form x^j, $b_1 x^j$, $b_2 x^j$ or $b_3 x^j$. It turns out that $H \not\subset \mathrm{Im}(f)$, a contradiction. ◻

From the preceeding, we know that the only group type to consider is Group type I). We

denote by $(e,0)$ the neutral element of $D_n \times \mathbb{Z}/2$, where $\mathbb{Z}/2$ is additively generated by 1.

For the action of the braid group on the set of Hurwitz vectors we make use of Lemma 2.1 in [CLP1].

Lemma 5.10. *Every Hurwitz vector of length r with elements in D_n of the form*

$$v = (v_1, ..., yx^a, yx^b, yx^c, ..., v_r)$$

is equivalent to $v' = (v_1, ..., yx^{a'}, yx^{a'}, yx^{c'}, ..., v_r)$ or $v'' = (v_1, ..., yx^{a'}, yx^{b'}, yx^{b'}, ..., v_r)$ via braid moves that only affect the triple (yx^a, yx^b, yx^c).

Lemma 5.11. *Classification of cover type I)*

*In this case the only admissible Hurwitz vector for n **odd** is*

$$v = ((y,1),(y,1),(yx,1),(yx,1),(e,1),(e,1)).$$

*For n **even** (n=2m) there are the following possibilities:*

$$v = ((y,1),(y,1),(yx,1),(yx,1),(e,1),(e,1)),$$

$$v = ((y,1),(yx^m,1),(yx,1),(yx,1),(x^m,1),(e,1)),$$

$$v = ((y,1),(yx^m,1),(yx^2,1),(yx^2,1),(x^m,1),(e,1)), m \textit{ odd.}$$

For $n = 2$ there are the following:

$$v = ((y,1),(y,1),(x,1),(x,1),(e,1),(e,1)),$$

$$v = ((y,1),(yx,1),(x,1),(x,1),(x,1),(e,1))$$

Proof. Since the cover $C/H \to \mathbb{P}^1$ branches in 6 points (cf. [MSSV]) we need a Hurwitz vector with second component equal to 1. So we have

$$v = ((y^{k_1}x^{l_1},1),(y^{k_2}x^{l_2},1),(y^{k_3}x^{l_3},1),(y^{k_4}x^{l_4},1),(y^{k_5}x^{l_5},1),(y^{k_6}x^{l_6},1))$$

The first observation is that the condition $< v >= G$ implies that there must exist j, s.t. $k_j = 1$. Therefore up to automorphism we can assume

$$v = ((y,1),(y^{k_2}x^{l_2},1),(y^{k_3}x^{l_3},1),(y^{k_4}x^{l_4},1),(y^{k_5}x^{l_5},1),(y^{k_6}x^{l_6},1))$$

We consider the two cases n odd and n even separately.

i) n odd: Not all k_j can be equal to 1. Otherwise we cannot generate the element $(y,0)$. Now the only element of order two of the form $(x^l,1)$ in G is $(e,1)$. So because of the product one condition v either looks like

$$v = ((y,1),(yx^{l_2},1),(yx^{l_3},1),(yx^{l_4},1),(e,1),(e,1))$$

or

$$v = ((y,1),(y,1),(e,1),(e,1),(e,1),(e,1)),$$

the latter being excluded, since $G \neq < v >$.

The product one condition gives $l_2 + l_4 \equiv l_3 \mod n$. The condition $< v >= G$ implies $\gcd(l_2,l_3,l_4,n) = \gcd(l_2,l_4,n) = 1$. Since the second factor $\mathbb{Z}/2$ of G is abelian, we can apply Lemma 5.10 to achieve that $l_3 = l_4$. Now v looks like

$$v = ((y,1),(yx^{l_2},1),(yx^{l_4},1),(yx^{l_4},1),(e,1),(e,1))$$

and again by product one we obtain $l_2 \equiv 0 \mod n$ and therefore $1 = \gcd(l_2,l_4,n) = \gcd(l_4,n)$.

So we can apply the automorphism $(x^{l_4},0) \mapsto (x,0), (y,0) \mapsto (y,0)$ to v and we can take

$$v = ((y,1),(y,1),(yx,1),(yx,1),(e,1),(e,1))$$

as a Hurwitz vector for the covering $C \to \mathbb{P}^1$.

ii) n even: Recall the general form:

$$v = ((y,1),(y^{k_2}x^{l_2},1),(y^{k_3}x^{l_3},1),(y^{k_4}x^{l_4},1),(y^{k_5}x^{l_5},1),(y^{k_6}x^{l_6},1))$$

Again, first we distinguish the possible Hurwitz vectors by the (even and positive) number of k_j that are equal to 1. We call the element $y^k x^l$ a reflection if $k \equiv 1(\mod 2)$.

In the current case there exists $m = n/2$, which gives the extra order 2 element $(x^m, 1) \in G$. As in the odd case, 6 reflections cannot occur. For the case of 2 reflections, assume, up to ordering,

$$v = ((y, 1), (yx^{l_2}, 1), (x^{l_3}, 1), (x^{l_4}, 1), (x^{l_5}, 1), (x^{l_6}, 1)).$$

As before, $(l_3, l_4, l_5, l_6) = (0, 0, 0, 0)$ is impossible. In the cases $(l_3, l_4, l_5, l_6) = (m, m, 0, 0)$ and $(l_3, l_4, l_5, l_6) = (m, m, m, m)$ we get $l_2 = 0$. In the first case we can only have $< v >= G$ if $n = 2$. Also in the second case we must have $n = 2$ but the elements $(y, 1)$ and $(x, 1)$ cannot generate G since the element $(e, 1)$ is missing. In the cases $(l_3, l_4, l_5, l_6) = (m, m, m, 0)$ and $(l_3, l_4, l_5, l_6) = (m, 0, 0, 0)$ we get $l_2 = m$ which also implies that $n = 2$. So if $n > 2$ these cases don't occur. The corresponding Hurwitz vectors are:

$$v = ((y, 1), (y, 1), (x, 1), (x, 1), (e, 1), (e, 1)),$$

$$v = ((y, 1), (yx, 1), (x, 1), (x, 1), (x, 1), (e, 1))$$

and

$$v = ((y, 1), (yx, 1), (x, 1), (e, 1), (e, 1), (e, 1)),$$

the third one being equivalent to the second one by an automorphism of G that fixes D_n.

Assume, for the case of 4 reflections, up to ordering

$$v = ((y, 1), (yx^{l_2}, 1), (yx^{l_3}, 1), (yx^{l_4}, 1), (x^{l_5}, 1), (x^{l_6}, 1)).$$

Here we have the 3 cases: $l_5 = l_6 = m$, $l_5 = l_6 = 0$ and $l_5 = m$, $l_6 = 0$.

In the first 2 cases from the product-one condition we get $l_2 + l_4 \equiv l_3 \mod n$. To generate G we must have $\gcd(l_2, l_3, l_4, n) = \gcd(l_2, l_4, n) = 1$.

Using Lemma 5.1 again, we arrive at

$$v = ((y, 1), (yx^{l_2}, 1), (yx^{l_4}, 1), (yx^{l_4}, 1), (x^m, 1), (x^m, 1))$$

resp.

$$v = ((y, 1), (yx^{l_2}, 1), (yx^{l_4}, 1), (yx^{l_4}, 1), (e, 1), (e, 1))$$

and so we get $l_2 \equiv 0 \mod n$. Now we have $\gcd(l_2, l_4, n) = \gcd(l_4, n) = 1$ and we can apply the automorphism $x^{l_4} \mapsto x, y \mapsto y$ to v to arrive at

$$v = ((y, 1), (y, 1), (yx, 1), (yx, 1), (x^m, 1), (x^m, 1))$$

resp.

$$v = ((y, 1), (y, 1), (yx, 1), (yx, 1), (e, 1), (e, 1)).$$

Using the morphism $(e, 1) \mapsto (x^m, 1), (y, 0) \mapsto (yx^{-m}, 0)$ we see that these two are equivalent.

It remains to consider the case $l_5 = m$ and $l_6 = 0$, i.e.

$$v = ((y, 1), (yx^{l_2}, 1), (yx^{l_3}, 1), (yx^{l_4}, 1), (x^m, 1), (e, 1)).$$

We apply Lemma 2.1, [CLP1] again and it follows $l_2 = m$. So we get

$$v = ((y, 1), (yx^m, 1), (yx^l, 1), (yx^l, 1), (x^m, 1), (e, 1))$$

where $\gcd(l, m) = 1$.

We have two sub cases, i.e. $\gcd(l, n) = 1$ and $\gcd(l, n) = 2$. In the first case we can use the automorphism $x^l \mapsto x, y \mapsto y$ to obtain

$$v = ((y, 1), (yx^m, 1), (yx, 1), (yx, 1), (x^m, 1), (e, 1)).$$

In the second case (where m must be odd) we can achieve

$$v = ((y, 1), (yx^m, 1), (yx^2, 1), (yx^2, 1), (x^m, 1), (e, 1)).$$

□

Lemma 5.12. *Classification of cover type II)*
Up to equivalence, the admissible f is given by the Hurwitz vector:
(1) $c_5 = 2$,

$$v = ((y, 1), (yx, 1), (yx, 1), (e, 1), (y, 0)),$$

(2) $c_5 > 2$,

$$v = ((y,1),(yx^{-1},1),(e,1),(e,1),(x,0)), c_5 = n,$$

$$v = ((y,1),(yx^{m-1},1),(x^m,1),(e,1),(x,0)), n = 2m, c_5 = n,$$

$$v = ((y,1),(yx^{m-2},1),(x^m,1),(e,1),(x^2,0)), n = 2m, m \textit{ is odd}, c_5 = m,$$

Proof. Assume we have an admissible $f : T(2,2,2,2,c_5) \to D_n \times \mathbb{Z}/2$.
we must have:

$$v := (f(\gamma_1), f(\gamma_2), f(\gamma_3), f(\gamma_4), f(\gamma_5)) = ((a_1,1),(a_2,1),(a_3,1),(a_4,1),(a_5,0))$$

There are two cases:
(1) $\underline{c_5 = 2}$.
As in the previous argument, we do the classification in terms of the number of reflections in $\{a_i\}$, which can be either 2 or 4.
(i) There are 2 reflections.
(a) a_5 is a reflection, W.L.O.G we can assume a_1 is another reflection, and $a_1 = yx^l, a_5 = y$. $a_2, a_3, a_4 \in \{e, x^{n/2}(\text{if n is even})\}$.
There are 4 cases (up to an order change): α) $(a_2,a_3,a_4) = (e,e,e)$, β) $(a_2,a_3,a_4) = (x^{n/2},e,e)$, γ) $(a_2,a_3,a_4) = (x^{n/2},x^{n/2},e)$, δ) $(a_2,a_3,a_4) = (x^{n/2},x^{n/2},x^{n/2})$.
Case α),δ) we get no admissible f since f can not be surjective.
For case β), γ) (where n is even) we get f is admissible $\iff n = 2$.
(b) a_5 is not a reflection, first we conclude that n must be even and $a_5 = x^{n/2}$. Using similar arguments as in a), one finds that

$$v = ((y,1),(yx^l,1),(a_3,1),(a_4,1),(x^{n/2},0)),\ a_3,a_4 \in \{e, x^{n/2}\}.$$

There are three cases, and one checks easily that in each case f is admissible if and only if $n = 2$.
(ii) There are 4 reflection.
a) a_5 is a reflection. W.L.O.G we assume

$$v = ((yx^{l_1},1),(yx^{l_2},1),(yx^{l_3},1),(a_4,1),(y,0)),\ a_4 \in \{e, x^{n/2}\textit{(if n is even)}\}.$$

Again we apply Lemma 5.1 so that we can assume $l_2 = l_3$. Since f is admissible, (using similar arguments as in the previous Lemma,) we have:
Case α) If $a_4 = e$, then $l_1 \equiv 0$ (n), $\gcd(l_2,n) = 1$. Under the automorphism $x^{l_2} \mapsto x, y \mapsto y$, we get

$$v \sim ((y,1),(yx,1),(yx,1),(e,1),(y,0)).$$

Case β) $n = 2m$ and $a_4 = x^m$. One gets $l_1 \equiv m$ $(2m)$, and $\gcd(l_2 - m, 2m) = 1$. Using the automorphism $x^{l_2-m} \mapsto x, y \mapsto y$, then we can achieve

$$v \sim ((yx^m, 1), (yx^{m+1}, 1), (yx^{m+1}, 1), (x^m, 1), (y, 0)).$$

Using the automorphism (of G): $(x, 0) \mapsto (x, 0)$, $(y, 0) \mapsto (y, 0)$, $(e, 1) \mapsto (x^m, 1)$, one finds that Case β) is equivalent to Case α).

b) a_5 is not a reflection.

In this case n must be even, and $v = ((y, 1), (yx^{l_2}, 1), (yx^{l_3}, 1), (yx^{l_4}, 1), (x^{n/2}, 0))$. It is easy to see that f can not be surjective since $(y, 0)$ is not contained in the image.

Up to now we have got all the admissible f's for the case $n = 2$. (Since $n = 2$ implies that $c_5 = 2$). One checks easily that they are equivalent to each other, since in this case G is abelian.

(2) $\underline{c_5 > 2}$.

a_5 must lie in the cyclic subgroup, say $a_5 = x^k$ ($k \neq \frac{n}{2}$ if n is even).

(i) There are 2 reflections, W.L.O.G. we assume

$$v = ((y, 1), (yx^l, 1), (a_3, 1), (a_4, 1), (x^k, 0)), \ a_3, a_4 \in \{e, x^{n/2}(if\ n\ is\ even)\}$$

There are 3 cases:

Case α) $(a_3, a_4) = (e, e)$.

We get $l + k \equiv 0 \pmod n$ and $\gcd(k, n) = 1$. Applying the automorphism $x^k \mapsto x, y \mapsto y$ we get

$$v \sim ((y, 1), (yx^{-1}, 1), (e, 1), (e, 1), (x, 0)).$$

Moreover we see that $c_5 = n$.

Case β) $n = 2m$ and $(a_3, a_4) = (x^m, e)$.

We get $l + k \equiv m \pmod{2m}$ and $\gcd(k, m) = 1$.

If $\gcd(k, n) = 1$ (which is the unique case if $2|m$),

$$v \sim ((y, 1), (yx^{m-1}, 1), (x^m, 1), (e, 1), (x, 0))$$

Here we find $c_5 = n$.

Otherwise $\gcd(k, n) = 2$ (which may happen only when $2 \nmid m$),

$$v \sim ((y, 1), (yx^{m-2}, 1), (x^m, 1), (e, 1), (x^2, 0))$$

and we have $c_5 = m$.

Case γ) $n = 2m$ and $(a_3, a_4) = (x^m, x^m)$.

We get $l + k \equiv 0\ (n)$ and $\gcd(k, n) = 1$.

$$v \sim ((y, 1), (yx^{-1}, 1), (x^m, 1), (x^m, 1), (x, 0)), c_5 = n$$

Using the automorphism $(x, 0) \mapsto (x, 0)$, $(y, 0) \mapsto (yx^{-m}, 0)$, $(e, 1) \mapsto (x^m, 1)$, one finds case γ) is equivalent to Case α).
(ii) There are 4 reflections.
One checks easily that f can not be surjective since $(y, 0) \notin \mathrm{Im}(f)$. □

Lemma 5.13. *Classification of cover type III-a)*
We have that $n = 2m$ and $d_4 = m$. Up to equivalence there is a unique admissible f given by the Hurwitz vector:

$$v = ((y, 1), (yx^{-1}, 1), (e, 1), (x, 1)).$$

Proof. Assume $f : T(2, 2, 2, 2d_4) \to D_n \times \mathbb{Z}/2$ is admissible.

$$v := (f(\gamma_1), f(\gamma_2), f(\gamma_3), f(\gamma_4)) = ((a_1, 1), (a_2, 1), (a_3, 1), (a_4, 1)).$$

$d_4 > 1 \Rightarrow a_5 = x^k$ ($k \neq n/2$ if n is $even$).
There can only be 2 reflections among a_1, a_2, a_3. W.L.O.G. we can assume

$$v = ((y, 1), (yx^l, 1), (a_3, 1), (x^k, 1)), a_3 \in \{e, n/2(if\ n\ is\ even)\}$$

Case a) $a_3 = e$.
We get $l + k \equiv 0 \pmod n$ and $\gcd(k, n) = 1$,

$$v \sim ((y, 1), (yx^{-1}, 1), (e, 1), (x, 1))$$

In this case $2d_4 = n$, it turns out that n must be even.
Case b) $n = 2m$ and $a_3 = x^m$.
We get $l + k \equiv m\ (2m)$ and $\gcd(l, n) = 1$,

$$v \sim ((y, 1), (yx, 1), (x^m, 1), (x^{m-1}, 1)).$$

Using the automorphism $(x, 0) \mapsto (x^{-1}, 0)$, $(y, 0) \mapsto (yx^{-m}, 0)$, $(e, 1) \mapsto (x^m, 1)$, we find that Case b) is equivalent to Case a). □

Lemma 5.14. *Classification of cover type III-b)*
We have that $c_3 = 2$ and $c_4 = n$. Up to equivalence there is a unique admissible f given by the

Hurwitz vector:

$$v = ((yx, 1), (e, 1), (y, 0), (x, 0)).$$

Proof. From Example 5.6 we see if that a type $III - b$) cover has group type 1), c_3 must be 2, combining with the proof of Corollary 3.2 one obtains that the case $(\delta_H, \delta_{H'}) = (1, 3)$ does not occur.

Let $f : T(2, 2, 2, c_4) \to D_n \times \mathbb{Z}/2$ be admissible. We must have

$$v := (f(\gamma_1), f(\gamma_2), f(\gamma_3), f(\gamma_4)) = ((a_1, 1), (a_2, 1), (a_3, 0), (a_4, 0))$$

Since $c_4 > 2$ we get $a_4 = x^k$. It is obvious that there are two (and only two) reflections among a_1, a_2, a_3.

(1) a_3 is not a reflection. n must be even (let $n = 2m$) and $a_3 = x^m$. W.L.O.G we assume

$$v = ((y, 1), (yx^l, 1), (x^m, 0), (x^k, 0)).$$

It is easy to see $(y, 0) \notin \mathrm{Im}(f)$, therefore in this case there is no admissible f.

(2) a_3 is a reflection. W.L.O.G we assume

$$v = ((yx^l, 1), (a_2, 1), (y, 0), (x^k, 0)), a_2 \in \{e, n/2(if\ n\ is\ even)\}.$$

(i) $a_2 = e$, we get $k \equiv l\ (n)$ and $\gcd(k, n) = 1$,

$$v \sim (yx, 1), (e, 1), (y, 0), (x, 0)), c_4 = n.$$

(ii) $n = 2m$ and $a_2 = x^m$, we get $k \equiv l + m\ (2m)$, $\gcd(k, n) = 1$,

$$v \sim (yx^{m+1}, 1), (x^m, 1), (y, 0), (x, 0)), c_4 = n.$$

Using the automorphism $(x, 0) \mapsto (x, 0)$, $(y, 0) \mapsto (y, 0)$, $(e, 1) \mapsto (x^m, 1)$, we see that Case (ii) is equivalent to Case (i). □

Remark 5.15. *If we drop the restriction on f_H, it is easy to check that the Hurwitz vectors in $III - a$) and $III - b$) are equivalent. (Consider the automorphism of G: $(x, 0) \mapsto (x, 1)$, $(y, 0) \mapsto (yx, 0)$, $(e, 1) \mapsto (e, 1)$)*

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Authors' Address:
Binru Li, Sascha Weigl
Lehrstuhl Mathematik VIII, Universität Bayreuth
Universitätsstraße 30, D-95447 Bayreuth
E-mail address:
binru.li@uni-bayreuth.de
sascha.weigl@uni-bayreuth.de